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# Measuring income inequality in social networks

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ORIGINAL RESEARCH

# Measuring income inequality in social networks

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## Abstract

We present a new index for measuring income inequality in networks. The index is based on income comparisons made by the members of a network who are linked with each other by direct social connections. To model the comparisons, we compose a measure of relative deprivation for networks. We base our new index on this measure. The index takes the form of a ratio: the network's aggregate level of relative deprivation divided by the aggregate level of the relative deprivation of a hypothetical network in which one member of the network receives all the income, and it is with this member that the other members of the network compare their incomes. We discuss the merits of this representation. We inquire how changes in the composition of a network affect the index. In addition, we show how the index accommodates specific network characteristics.

*Keywords*: Income inequality in networks; Relative deprivation in networks; An index of income inequality in networks; Compositional changes of networks

*JEL classification*: D31; D63; I31; L14

## 1. Introduction

Because by its very nature a network is a social architecture that is different from a population at large, the usual way in which income inequality is measured in a population cannot seamlessly be applied to networks. This is not all that surprising because the protocols of income comparisons differ. Whereas in a population a standard measure of income inequality such as the Gini coefficient counts all the income comparisons between pairs (an approach that can be supported because every member is "connected" with every other member), in a network there is no such uniformity: the group of people with whom an

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individual is connected, or with whom he compares his income, is specific to that individual and, typically, this group is smaller than the population at large. Individuals have comparison groups, consisting of other individuals with whom they compare their incomes, and those comparators have their own comparison groups, and so on; the loops need not close or overlap. This chain of comparisons is at the heart of the structure of a network.

When people change their jobs and the places in which they live, the social matrices of their colleagues, acquaintances, and friends change, as does the set of their links. The structure of a network affects the choices of its members. This influence has been noted, for example, by Calvo-Armengol and Jackson (2004) in an employment model, and by Tsvetkova et al. (2018) in an experimental study. Melamed et al. (2022) have shown that wealth inequality induces individuals to form new connections, resulting in a network structural change. A recurrent theme in the existing network literature is that individuals forge and discard links so as to improve their overall position (consult, for example, Goyal and Vega-Redondo, 2007). The position of an individual in a network and the architecture of the network influence the individual's perception of inequality.

The study of networks is guided by the observation that, as underscored in a large literature on social networks, individuals typically interact with other individuals in small groups (Ghiglino and Goyal, 2010; Jackson, 2010; Goyal, 2011; Immorlica et al., 2017; Jackson et al., 2020; and others). This characteristic has long been emphasized in studies of human evolution, a context in which the "Dunbar's number" (Dunbar, 1992) comes to mind: there is a cognitive limit to the number of people with whom an individual can maintain stable social relationships and, presumably, bring into his orbit of comparisons. That individuals compare their incomes only with their network neighbors can have critical implications for their behavior (see Calvo-Armengol and Jackson, 2004; Melamed et al., 2022). The network neighbors of an individual are the members of the network with whom the individual is connected.

Several indices of inequality already feature in studies of social networks. These indices are of two types: those that are based on differences in social connections (Horvath and Zhang, 2018), and those that are based on income (payoff) comparisons. Among the indices belonging to the second, better populated, type are the ratio of the highest income to the lowest income (Gagnon and Goyal, 2017); Atkinson's class of inequality measures (Ambrus and Elliott, 2021); and second-order stochastic dominance of income (consumption) distributions (Bourles et al., 2017). Arguably, the most popular index of income inequality featuring in studies of social networks is the Gini coefficient (Cavalcanti and Giannitsarou, 2017; Tsvetkova et al., 2018; Plotnikova and Ulceluse, 2022; and Toth et al., 2021).

The existing indices of inequality in networks do not utilize the complete data of a network's income distribution and of the same network's architecture of connections, although both these factors impact on the level of the income inequality experienced by members of the network. For example, consider an individual who is connected (compares his income) with (the income of) a much richer individual. If the first individual severs this link, then, naturally, he will experience a considerably lower level of income inequality. The same argument applies when his income increases measurably while the incomes of others do not. Therefore, both the network architecture and income distribution matter to an individual. A proper index of income inequality should cater for these dependencies.

In this paper we construct an index of income inequality that is tailor-made for networks. We base this index on a measure of relative deprivation. We say that an individual is relatively deprived when his income is lower than the incomes of his comparators (these are the individuals with whose incomes the individual compares his own income). We compose a measure of relative deprivation for networks. Following Stark et al. (2017), we show that the index in Yitzhaki (1979), which is based on Atkinson's index (Atkinson, 1970), can serve as a measure of relative deprivation adjusted to networks. We aggregate the levels of relative deprivation experienced by the members of a network to obtain the network's aggregate level of relative deprivation, which we refer to as total relative deprivation, *TRD*. We show that a fitting index of income inequality in networks is the following ratio: *TRD* divided by a "variant" of *TRD* calculated for a network in which one person with whose income the other members of the network compare their incomes receives all the income of the network.

We inquire how our new index "behaves" in a dynamic setting, that is, we consider situations in which, while the incomes of the individuals in a network are held constant, the individuals are repositioned in the network. We formulate conditions under which, when a new link is added, the value of the index decreases / increases, and conditions under which, when an existing link is severed, the value of the index decreases / increases. These conditions have the nice feature that they yield an association between a decrease of the level of the index of income inequality in networks and an increase in each of three network characteristics: the average degree of a network, the network density, and the clustering coefficient. (In Section 2 we introduce and explain these terms.) In addition, we show that holding constant the incomes of the individuals, the index attains its maximal value when the network architecture is such that the individuals who are relatively deprived form an independent set. The "behavior" of the index in a changing environment is of natural interest in studies of networks, both because quite often the structure of the links (relations, comparisons) in a social network is dynamic rather than static, and because of a preference to reduce inequality while keeping incomes intact.

We proceed as follows. In Section 2, we present notation and characterizations of networks. In Section 3, we compose a measure of relative deprivation for networks, and we highlight properties that differentiate this measure from the standard measure of relative deprivation. In addition, we introduce a measure of total relative deprivation, *TRD*. Drawing on *TRD*, in Section 4 we construct our index of income inequality in networks, and we list properties of this index. In Section 5, we investigate how the index changes when the architecture of the network changes. In Section 6, we conclude.

## 2. A network building blocks: Notation, and characterizations

In this section we present several terms that will be needed for our study of networks. Throughout this paper we use a number of terms that feature in the standard vocabulary of network research. For ease of reference, we briefly explain these terms. (For definitions, usages, and examples of terms, Jackson (2010), among others, can gainfully be consulted.)

Drawing on the standard terminology of network topology, we define a network as a pair $\mathcal{N} = (V, E)$ such that $V = \{1, 2, ..., n\}$, where $n \geq 3$, is a set of nodes that corresponds to a set of individuals, and $E \subset V \times V$ is a set of links. We denote by $ij$ the link that connects nodes $i \in V$ and $j \in V$.[1] We make two assumptions.

Assumption 1. The network $\mathcal{N} = (V, E)$ has no self-loops (no links that connect a node with itself), that is, there are no $ii$ or $jj$ links.

[1] The networks studied in this paper are undirected networks, meaning networks in which the connections between the individuals are mutual. However, our results can easily be adapted to directed networks, meaning networks in which the connections between the individuals are directional: the connections are not mutual.

Assumption 2. Between any two nodes of $\mathcal{N} = (V, E)$ there is at most one link.[2]

A path between two nodes $i, j \in V$ is a sequence of links $i_1 i_2, i_2 i_3, \ldots, i_{k-2} i_{k-1}, i_{k-1} i_k \in E$, for a natural number $k > 1$, such that $i_1 = i$ and $i_k = j$. By $\text{dist}(i, j)$ we denote the distance between nodes $i, j \in V$, meaning the length of the shortest path connecting $i$ and $j$. When no path connects $i$ and $j$, we set $\text{dist}(i, j) = \infty$. A set $W \subset V$ of nodes is an independent set if there are no links between the nodes of $W$.

We refer to $\bar{\mathcal{N}} = \left(\bar{V}, \bar{E}\right)$ as a sub-network of $\mathcal{N} = (V, E)$ when $\bar{V} \subset V$ and $\bar{E} \subset E$ is the set of links between the nodes of $\bar{V}$. In this case, we write $\bar{\mathcal{N}} \subset \mathcal{N}$. A sub-network $\mathcal{C} = (V_{\mathcal{C}}, E_{\mathcal{C}})$, $\mathcal{C} \subset \mathcal{N}$, is a connected component of $\mathcal{N} = (V, E)$ when the following two properties hold.

1. Between any two nodes of $V_{\mathcal{C}}$ there is a path that consists of $E_{\mathcal{C}}$ links.

2. There is no other sub-network $\bar{\mathcal{N}} \subset \mathcal{N}$ that satisfies the preceding property, and such that $\mathcal{C} \subset \bar{\mathcal{N}}$.

A network $\mathcal{N} = (V, E)$ can be represented as the sum of its disjoint connected components such that there are no links between nodes from different components. In the specific case in which a network is complete, there is only one connected component comprising the entire network; and in another specific case of an empty network (a network without links), each node forms a connected component.

We consider a network $\mathcal{N} = (V, E)$ of $n \geq 3$ individuals such that the income of individual $i \in V$ is $x_i \geq 0$, and such that there is at least one individual whose income is strictly positive. We denote the income distribution of $V$ by $X$, that is, $X = (x_1, x_2, \ldots, x_n)$. To simplify notation, we occasionally refer to $\mathcal{N} = (V, E)$ as $\mathcal{N}$. The structure of $(\mathcal{N}, X)$ is such that there is a link $ij \in E$ if individual $i$ compares his income with the income of individual $j$. For each node $i \in V$, let $N_i \equiv \{j \in V : ij \in E\}$ be the comparison group of individual $i$, that is, the nodes with which individual $i$ has a link. These nodes constitute the set of $i$'s neighbors in the network. (In line with the conventional terminology of networks, here too the term "neighbors" means the set of nodes with which individual $i$ is connected, so the term means the individual's comparison group.) We denote by $d_i = |N_i|$ the degree of a node $i$, that is, the number of links that $i$ has with other nodes, and by $V^* \equiv \{i \in V : d_i > 0\}$ the set of nodes that are connected with at least one other node.

We introduce and explain three features of networks, or network characteristics: the average degree of a network; the network density; and the clustering coefficient. Let $\mathcal{N} = (V, E)$ be a network of $n \geq 3$ nodes.

*The average degree of a network*: this is the sum of the degrees of the nodes divided by the number of the nodes.

*The network density*: this is the number of connections in $\mathcal{N}$ divided by the number of connections in a complete network of $n$ nodes. Because in a network the sum of the degrees is equal to twice the number of the connections, we can express the network density as

$$\frac{|E|}{\frac{n(n-1)}{2}} = \frac{\frac{2}{n}|E|}{n-1} = \frac{\frac{1}{n}\sum_{i \in V} d_i}{n-1},$$

that is, the network density is the average degree of a network divided by $n - 1$.

[2] When referring to directed networks, we consider a two-way link as one link.

*The clustering coefficient*: this is the fraction of fully connected triples of nodes out of the potential of triples in which at least two links are present. Formally, the clustering coefficient is

$$\frac{\sum_i \left|\{jk \in E : j \neq k,\, j \in N_i,\, k \in N_i\}\right|}{\sum_i \left|\{jk \in V \times V : j \neq k,\, j \in N_i,\, k \in N_i\}\right|} \quad \text{if } \exists i \in V : d_i \geq 2,$$

and 0 otherwise, where $N_i$ is the comparison group of individual $i$. Put heuristically, the clustering coefficient indicates how often two comparators of an individual are also comparators of each other.

## 3. Relative deprivation in networks

In this section we formulate the relative deprivation of an individual in a network, and the aggregate or the total relative deprivation in a network.

### 3.1 A measure of the relative deprivation in a network

Our measure of relative deprivation in networks considers the neighborhood of an individual as his comparison group, and computes the individual's average income shortfall which results from comparisons of his income with the incomes of his neighbors. We show how the position that an individual occupies in a network affects the level of his relative deprivation.

In network $\mathcal{N}$ of $n$ individuals, $n \geq 3$, let the income distribution be $X = (x_1, \ldots, x_n)$; we refer to the pair $(\mathcal{N}, X)$ as an "income network:" that is, an income network is a network associated with an income distribution. We construct a measure of relative deprivation of a member of an income network by drawing on the axioms presented in Stark et al. (2017). We adjust these axioms to fit the setting of a network.

**Observation 1**. In income network $(\mathcal{N}, X)$, the relative deprivation of individual $i$ whose comparison group is $N_i$ is given by

$$RD(i, N_i) = \begin{cases} \dfrac{1}{d_i} \displaystyle\sum_{j \in N_i} \max\{x_j - x_i, 0\} & \text{if } d_i > 0, \\ 0 & \text{if } d_i = 0. \end{cases} \tag{1}$$

Observation 1 follows from the fact that the individual's neighbors in the network constitute the individual's comparison group. The set of axioms presented in Stark et al. (2017) can be adapted to obtain (1). Application of the adapted set yields a one-parameter family of measures of relative deprivation of an individual. The measure $RD(i, N_i)$ in (1) can be obtained by supplementing the Stark et al. (2017) axioms with the Transfer Property, which says that a top-down or a bottom-up transfer of some positive income between two individuals (comparators of individual $i$) who are wealthier than individual $i$ does not change $RD(i, N_i)$, provided that following the transfer, the transferor does not become poorer than individual $i$.

Because in making income comparisons the relative deprivation of an individual is a local measure, we see that when calculating (1), only the income distribution and the number of neighbors of the individual (the size of his comparison group) matter. The measure of relative deprivation in (1) informs us that individual $i$ compares his income with the incomes of the members of his comparison group $N_i$, and that when individual $i$ has no connections, he does not experience relative deprivation.

In comparison with a standard formulation of relative deprivation, (1) is more intricate: the structure of the network matters as does the position of the individual (whose level of relative deprivation is measured) in the network. We illustrate this difference with the help of an example.

*Example 1*. We consider a network of four individuals, as displayed in Figure 1. The numbers in the nodes are the names of the individuals.

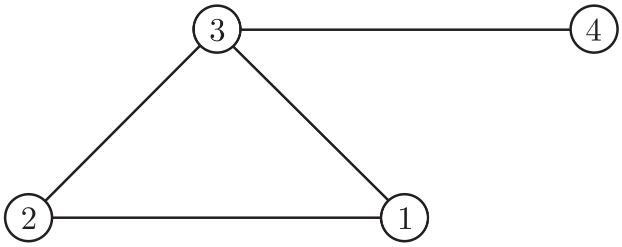


**Figure 1.** A network of four individuals.

We endow the individuals with the following income distribution $X = (x_1, x_2, x_3, x_4)$: $x_1 = 2,\ x_2 = 1,\ x_3 = 2,$ and $x_4 = 8$. Then only individuals 2 and 3 experience relative deprivation. In spite of individual 2 having the lowest income, his relative deprivation is not the highest. The level of relative deprivation of individual 3, which is $RD(3, \{1, 2, 4\}) = \frac{1}{3}(8 - 2) = 2$, is higher than the level of relative deprivation of individual 2, which is $RD(2, \{1, 3\}) = \frac{1}{2}(2 \cdot (2 - 1)) = 1$.

## 3.2 Total relative deprivation, *TRD*, in networks

In order to introduce the aggregate or the total relative deprivation, *TRD*, in a network, we formulate the following axiom.

**Additive Decomposition axiom**. Let $(\mathcal{N}, X)$ be an income network where $\mathcal{N} = (V, E)$. Equipping the network $\mathcal{N}$ with an additional node $l$ (and its connections), we obtain a new network $\hat{\mathcal{N}} = \left(\hat{V}, \hat{E}\right)$, where $\hat{V} = V \cup \{l\}$ and $\hat{E} = E \cup \bigcup_{j \in N_l} jl$. Then $TRD\left(\hat{\mathcal{N}}, X \cup \{x_l\}\right) = TRD\left(\hat{\mathcal{N}} \backslash \{l\}, X \cup \{x_l\}\right) + RD(l, N_l)$.[3]

Drawing on this axiom, we see that computing the *TRD* of a network equipped with an additional node requires updating the levels of relative deprivation of the neighbors of the node, and of adding the level of relative deprivation of the node.

[3]The $TRD\left(\hat{\mathcal{N}} \backslash \{l\}, X \cup \{x_l\}\right)$ term does not include the level of relative deprivation of individual $l$, while it includes the impact that the income of this individual has on the levels of relative deprivation of his neighbors in network $\hat{\mathcal{N}}$.

**Observation 2**. Let the Additive Decomposition axiom hold. Then, the total relative deprivation of income network $(\mathcal{N}, X)$ is equal to the sum of the levels of relative deprivation of the members of the network:

$$TRD(\mathcal{N}, X) = \sum_{i=1}^{n} RD(i, N_i).$$

In addition, given the measure of relative deprivation in (1),

$$TRD(\mathcal{N}, X) = \sum_{i \in V^*} \left( \frac{1}{d_i} \sum_{j \in N_i} \max\{x_j - x_i, 0\} \right). \tag{2}$$

It is easy to see that Observation 2 follows from an iterative application of the Additive Decomposition axiom and of Observation 1.

Because the individuals represented by isolated nodes are not relatively deprived, such nodes do not affect $TRD(\mathcal{N}, X)$. Therefore, the summation in (2) is applied to the nodes that have at least one link.

We next seek to identify a network architecture and an income distribution that cause *TRD* in (2) to reach its maximal level. Presumably, what causes *TRD* to be that high is of concern to a social planner who may want to know what to avoid most.

Let $\Omega_n$ be the income network of $n$ individuals, $n \geq 3$, that results in the highest level of $TRD(\mathcal{N}, X)$. To reach this level, the levels of relative deprivation of essentially all the members of $\mathcal{N}$ have to be maximal. From formula (1) of relative deprivation, we infer that such a result could be obtained by maximizing the income shortfall of every individual, except for one individual who is the richest, and by redesigning the network architecture in such a way that the individuals who experience income shortfalls are connected only with the richest individual. Consequently, income network $\Omega_n$ has to be a star network of $n$ individuals, $n \geq 3$, where the individual at the center of the star receives all the income of the network, $y$, that is, $y = \sum_{i=1}^{n} x_i$, and every other individual is left with no income.[4] Among the income networks that have the same number of nodes and the same aggregate income as does $\Omega_n$, the total relative deprivation attains its maximal value at $\Omega_n$. Using (2), we express the total relative deprivation of $\Omega_n$ as

$$TRD(\Omega_n) = \sum_{j \in V \setminus \{c\}} \frac{1}{1} (y - 0) = (n - 1)y, \tag{3}$$

where $c$ denotes the individual who is represented by the node at the center of the star.

## 4. An index of income inequality in networks

We now have in hand the components needed to construct our index of income inequality in networks. This index, $\Gamma(\mathcal{N}, X)$, of an income network $(\mathcal{N}, X)$ of $n$ individuals, $n \geq 3$, is a

[4] A star network is a network in which one node (the central node) has connections to all other nodes, while every other node has a connection only to the central node.

ratio: the *TRD* of the actual income network divided by the *TRD* of the hypothetical income network $\Omega_n$:

$$\Gamma(\mathcal{N}, X) \equiv \frac{TRD(\mathcal{N}, X)}{TRD(\Omega_n)} = \frac{\sum_{i=1}^{n} RD(i, N_i)}{(n-1)y} = \frac{\sum_{i \in V^*} \left( \frac{1}{d_i} \sum_{j \in N_i} \max\{x_j - x_i, 0\} \right)}{(n-1) \sum_{i=1}^{n} x_i}. \tag{4}$$

**Proposition 1**. Let $(\mathcal{N}, X)$ be an income network of $n$ individuals, $n \geq 3$. Then:
1. $0 \leq \Gamma(\mathcal{N}, X) \leq 1$.
2. $\Gamma(\mathcal{N}, X) = 0$ if and only if in every connected component of the network the component's income is divided equally between the component's members.
3. $\Gamma(\mathcal{N}, X) = 1$ if and only if - a relabeling of the nodes notwithstanding - the income network $(\mathcal{N}, X)$ is identical to the income network $\Omega_n$.
4. In a complete income network $(\mathcal{N}_C, X)$, the Gini coefficient and the index $\Gamma(\mathcal{N}_C, X)$ can be obtained from each other by means of rescaling.

**Proof**. The proof is in the Appendix.

The index (4) of income inequality in networks obtains the minimal value if and only if the members of the given connected component have the same income, and it obtains the maximal value if and only if the network is a star network of any number of individuals where only the individual at the center of the star receives income.

We present an example which shows how the index (4) can be used to rank by their levels of inequality two networks that have the same number of members and the same income distributions, yet differ in their architectures.

*Example 2*. Figures 2A and 2B portray networks $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, and the numbers in the nodes are the names of the individuals.

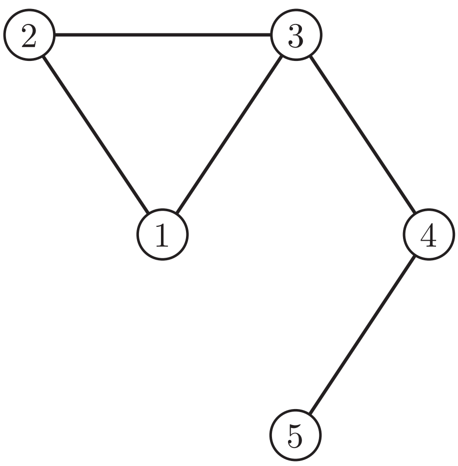


**Figure 2A.** Network $\mathcal{N}_1$.

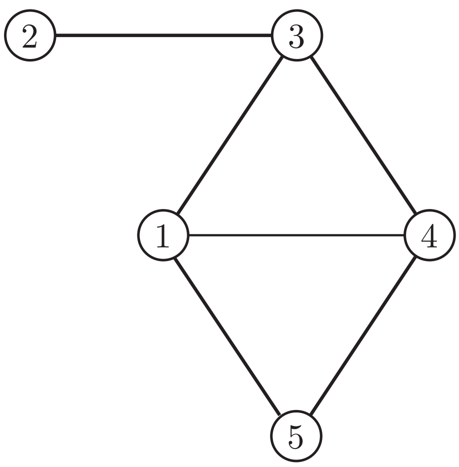


**Figure 2B.** Network $\mathcal{N}_2$.

We let the two networks have the same income distribution $X = (x_1, \ldots, x_5)$ where $x_1 = x_3 = 1$, individual 2 has income 9, that is, $x_2 = 9$, and $x_4 = x_5 = 2$. Let $N_i^j$ denote the comparison group of individual $i = 1, ..., 5$ in network $\mathcal{N}_j$, $j = 1, 2$. Then, in both networks, only individuals 1 and 3 are relatively deprived. We obtain

$$\Gamma(\mathcal{N}_1, X) = \frac{\sum_{i=1}^{5} RD(i, N_i^1)}{4\sum_{i=1}^{5} x_i} = \frac{\frac{1}{2}(9-1) + \frac{1}{3}((9-1)+(2-1))}{4(2\cdot 1 + 2\cdot 2 + 9)} \approx 0.1167,$$

and

$$\Gamma(\mathcal{N}_2, X) = \frac{\sum_{i=1}^{5} RD(i, N_i^2)}{4\sum_{i=1}^{5} x_i} = \frac{\frac{1}{3}(2(2-1)) + \frac{1}{3}((9-1)+(2-1))}{4(2\cdot 1 + 2\cdot 2 + 9)} \approx 0.0611.$$

Thus, $\Gamma(\mathcal{N}_1, X) > \Gamma(\mathcal{N}_2, X)$. This difference arises because individual 1 experiences lower relative deprivation in $\mathcal{N}_2$ than in $\mathcal{N}_1$ (there is no difference in the relative deprivation of individual 3). In other words, with income distribution $X = (x_1, ..., x_5)$, it is better for individual 1 to sever the connection with the richest neighbor (individual 2) and, instead, establish connections with individuals 4 and 5. In Section 5, we study in detail the topics of adding and severing links.

We next ascertain for which income distribution the index of income inequality in networks is maximized, conditional on both the architecture of the network and the income of the network held constant.

As a preliminary, we note that when individual $i \in V$ receives all the income of the network, $y$, the level of relative deprivation experienced by his comparator $j \in N_i$ is $RD(j, N_j) = \frac{1}{d_j}y$. Obviously, individuals who are not comparators of individual $i$ do not experience relative deprivation. In this case, $TRD(\mathcal{N}, X) = y\sum_{j\in N_i}\frac{1}{d_j}$. This expression varies with the choice of the individual who has income $y$, and it attains its maximal value when $\sum_{j\in N_i}\frac{1}{d_j}$ is maximal.

**Claim 1**. Let $y > 0$ be the aggregate income of network $\mathcal{N} = (V, E)$, and let $I$ be the set of all the individuals for whom the term $\sum_{j\in N_i}\frac{1}{d_j}$ is maximal. We then choose any nonempty independent set $J \subset I$. Then, the distribution of the network's income $y$, where $y$ is divided (in any manner) between the members of the set $J$, maximizes the index $\Gamma(\mathcal{N}, X)$. In particular, if $|I| = 1$, then $\Gamma(\mathcal{N}, X)$ records its maximal value when this one member of $I$ receives all the income of the network.

**Proof**. The proof is in the Appendix.

From Claim 1, we see that when the architecture and the aggregate income of a network are held constant, identifying an income distribution that maximizes the index of income inequality in networks is equivalent to identifying the individuals to whom the network's income is allocated and for whom $\sum_{j \in N_i} \frac{1}{d_j}$ attains the maximal value. When the set $J$ consists of several individuals, maximization of the index $\Gamma(\mathcal{N}, X)$ can be obtained by dividing the network's income between them in any manner. Here, again, we see how different the case of an arbitrary social network is from the case of a complete network, noting that in a complete network the choice of the individuals to whom all the income is to be given so as to maximize the income inequality is arbitrary.

In concluding this section, we comment briefly on the applicability of the Pigou-Dalton principle to networks. The reasons for alluding to this topic are as follows. The Pigou-Dalton transfer principle is a natural property of an inequality measure: a top-down transfer decreases inequality, a bottom-up transfer increases inequality. This is a fundamental characteristic of inequality measures such as the Gini coefficient and the Theil coefficient. Nevertheless, when we consider the Pigou-Dalton principle, we implicitly assume that an individual's perception of inequality is influenced by the income of everyone else in the population, so that every individual compares his income with the incomes of all members of the population. This quite natural assumption for a complete network cannot, however, be applied to other networks. For example, considering a top-down transfer, the transferor can have no individuals in his comparison group poorer than he is, while the transferee may have individuals in his comparison group who are poorer than he is. Then the top-down transfer can actually increase the inequality of the network. Therefore, the Pigou-Dalton transfer principle should not be expected to be satisfied for an inequality measure in a network. The following example elucidates.

*Example 3*. Consider a star network of six individuals. Assume that one of the peripheral individuals has income 4, and that the income of each of the other individuals is 0. In this case, only the central individual (the individual at the center of the star) is relatively deprived, and the level of inequality, as measured by our index $\Gamma(\mathcal{N}, X)$, is $\frac{\frac{1}{5}(4-0)}{5 \cdot 4} = \frac{1}{25} = 0.04$. After a transfer of 2 units of income from the peripheral individual to the central individual, each of the individuals whose income is 0 experiences relative deprivation, and the level of inequality in the network increases (tenfold) to $\frac{4 \cdot \frac{1}{1}(2-0)}{5 \cdot 4} = \frac{2}{5} = 0.4$.

Example 3 shows that the Pigou-Dalton principle need not hold in networks. However, the setting of Example 3 is somewhat unique in that the transfer is from an individual who has only one neighbor, whereas the transferee has many neighbors. In the following claim, we provide a sufficient condition under which the Pigou-Dalton principle does hold in networks.

**Claim 2**. Consider an income network $(\mathcal{N}, X)$ of $n$ individuals, $n \geq 3$. We assume a top-down (respectively, a bottom-up) rank-preserving transfer of income from individual $i$ to individual $k$. Then, the Pigou-Dalton principle holds, that is, the level of income inequality decreases (respectively, increases) when the following inequality is satisfied:

$$\sum_{\substack{l \in N_k \\ x_l > x_k}} \frac{1}{d_k} - \sum_{\substack{l \in N_k \\ x_l < x_k}} \frac{1}{d_l} \underset{(<)}{>} \sum_{\substack{j \in N_i \\ x_j > x_i}} \frac{1}{d_i} - \sum_{\substack{j \in N_i \\ x_j < x_i}} \frac{1}{d_j}.$$

**Proof**. The proof is in the Appendix.

Crudely speaking, Claim 2 states that the Pigou-Dalton principle holds for networks when (in the case of a top-down transfer) the transferor occupies a high position in the income hierarchy of his neighborhood and has many neighbors poorer than himself who themselves have small numbers of neighbors, while the transferee occupies a low position in the income hierarchy of his neighborhood and has a small number of neighbors poorer than himself who themselves have many neighbors.[5]

## 5. Tinkering with the network architecture

In this section, we supplement our analysis with a dynamic aspect. After all, networks expand and shrink, adjust and restructure. These revisions raise three questions regarding the architecture of networks and the positions occupied by their members.

1. Revision of the network structure. Under which conditions does adding a new link or severing an existing link increase the index of income inequality in networks, and under which conditions does it decrease the index? While not surprising, it is informative that changes of the level of the index $\Gamma(\mathcal{N}, X)$ are related to changes of the comparison groups and of the levels of relative deprivation of the individuals.

2. Network characteristics. How does the index of income inequality accommodate network characteristics? We consider three network characteristics that were introduced in Section 2: the average degree of a network, the network density, and the clustering coefficient. We formulate a condition under which an increase of each of these characteristics decreases the level of the index. We also show that shifting the position of the wealthiest individual to a more central position unequivocally increases the level of inequality in the network.

3. Configurations that maximize the index of income inequality in networks. When the individuals' incomes are fixed and the individuals can be repositioned in the network, which placement of the individuals maximizes the index $\Gamma(\mathcal{N}, X)$? As already noted in Section 3, this question can be assumed to be of interest to a social planner who might be particularly hostile to a network architecture that yields the highest level of inequality. The question is closely related, for instance, to a study by Kets et al. (2011), who explore the manner in which the structure of a social network constrains the network's level of inequality.[6]

To the preceding three questions we dedicate, respectively, Subsections 5.1, 5.2, and 5.3.

### 5.1 Revision of the network structure: Formation or severance of a link

An interaction between individuals who initially were separate can be modeled as the formation of a link between the individuals. And changes in the situation of individuals (in the real world, such changes can concern, for example, the place or the type of employment,

[5] This is not the first or the only constellation in which the Pigou-Dalton transfer principle fails to hold. Stark et al. (2018) show that a rich-to-poor transfer can induce a response in the individuals' behaviors which actually exacerbates, rather than reduces, income inequality. Stark et al. consider a stylized production economy (an "artisan economy") in which individuals produce a single consumption good. In such a constellation, a marginal rank-preserving transfer from a richer individual to a poorer individual can lead to exacerbated inequality. A Pigou-Dalton transfer from a richer individual to a poorer individual weakens the latter's incentive to work hard because the income "deprivation" experienced by the poorer individual is reduced. This scaling back of effort arises because, fundamentally, the benefits in terms of income when the poorer individual exerts effort are of two types: income as a means of enabling consumption, and income as a means of escaping an excessively low relative income.

[6] Despite different methods of measuring income inequalities, the results both of Kets et al. (2011) and ours rely crucially on independent sets.

and migration) can result in the severance of links. The formation and dissolution of links form a basis for network formation models (Jackson and Wolinsky, 1996; Bala and Goyal, 2000; Watts, 2001; Hellmann and Staudigl, 2014; Chen et al., 2020). Intuitively, a stability of a network configuration is reached when the formation of a new link or the severance of an existing link does not benefit any member of the network.

To begin with, we note that the formation / severance of a link that connects network individuals $i$ and $j$ affects the levels of relative deprivation only of these individuals. This observation follows from the definition of relative deprivation in (1).

We consider network $\mathcal{N} = (V, E)$ and two of its members: individual $i$ whose comparison group is $N_i$, and individual $j$ whose comparison group is $N_j$. Without loss of generality, we assume that the individuals' incomes satisfy $x_j \geq x_i$. The formation / severance of a link between these two individuals changes their comparison groups: we denote the comparison groups of individuals $i$ and $j$ after the formation of the link $ij$ as $N_i \cup \{j\}$ and $N_j \cup \{i\}$, and the comparison groups after the severance of the link $ij$ as $N_i \backslash \{j\}$ and $N_j \backslash \{i\}$, respectively.

We next describe the changes in the levels of relative deprivation of individuals $i$ and $j$ resulting from the formation / severance of the link $ij$.

**Observation 3**.
1. Assume that individuals $i$ and $j$ are not connected. Then, the formation of the link $ij$ has the following implications for the levels of relative deprivation of these individuals:

$$RD(i, N_i \cup \{j\}) - RD(i, N_i) = \frac{1}{d_i + 1}\Big[(x_j - x_i) - RD(i, N_i)\Big], \tag{5}$$

and

$$RD(j, N_j \cup \{i\}) - RD(j, N_j) = -\frac{1}{d_j + 1} RD(j, N_j). \tag{6}$$

2. Assume that individuals $i$ and $j$ are connected. Then, severance of the link $ij$ has the following consequences for the levels of relative deprivation of these individuals:

$$RD(i, N_i \backslash \{j\}) - RD(i, N_i) = \begin{cases} \frac{1}{d_i - 1}\Big[RD(i, N_i) - (x_j - x_i)\Big] & \text{if } d_i > 1, \\ -(x_j - x_i) & \text{if } d_i = 1, \end{cases} \tag{7}$$

and

$$RD(j, N_j \backslash \{i\}) - RD(j, N_j) = \begin{cases} \frac{1}{d_j - 1} RD(j, N_j) & \text{if } d_j > 1, \\ 0 & \text{if } d_j = 1. \end{cases} \tag{8}$$

To ease the reading of equations (5) and (7), we remark that equation (5) is obtained from the following sequence of equalities:

$$RD(i, N_i \cup \{j\}) - RD(i, N_i) = \frac{1}{d_i+1} \sum_{k \in N_i \cup \{j\}} \max\{x_k - x_i, 0\} - \frac{1}{d_i} \sum_{k \in N_i} \max\{x_k - x_i, 0\}$$

$$= \frac{1}{d_i+1}(x_j - x_i) + \frac{1}{d_i+1} \sum_{k \in N_i} \max\{x_k - x_i, 0\} - \frac{1}{d_i} \sum_{k \in N_i} \max\{x_k - x_i, 0\}$$

$$= \frac{1}{d_i+1}(x_j - x_i) - \left(\frac{1}{d_i} - \frac{1}{d_i+1}\right) \sum_{k \in N_i} \max\{x_k - x_i, 0\}$$

$$= \frac{1}{d_i+1}(x_j - x_i) - \frac{1}{(d_i+1)d_i} \sum_{k \in N_i} \max\{x_k - x_i, 0\}$$

$$= \frac{1}{d_i+1}\left[(x_j - x_i) - RD(i, N_i)\right].$$

And equation (7) is obtained from the following transformations:

$$RD(i, N_i \backslash \{j\}) - RD(i, N_i) = \frac{1}{d_i-1} \sum_{k \in N_i \backslash \{j\}} \max\{x_k - x_i, 0\} - \frac{1}{d_i} \sum_{k \in N_i} \max\{x_k - x_i, 0\}$$

$$= \frac{1}{d_i-1} \sum_{k \in N_i} \max\{x_k - x_i, 0\} - \frac{1}{d_i} \sum_{k \in N_i} \max\{x_k - x_i, 0\} - \frac{1}{d_i-1}(x_j - x_i)$$

$$= \left(\frac{1}{d_i-1} - \frac{1}{d_i}\right) \sum_{k \in N_i} \max\{x_k - x_i, 0\} - \frac{1}{d_i-1}(x_j - x_i)$$

$$= \frac{1}{(d_i-1)d_i} \sum_{k \in N_i} \max\{x_k - x_i, 0\} - \frac{1}{d_i-1}(x_j - x_i)$$

$$= \frac{1}{d_i-1}\left[RD(i, N_i) - (x_j - x_i)\right].$$

Needless to add, equations (6) and (8) are obtained in a manner that is similar to the manner in which equations (5) and (7) were obtained, respectively.

From (5) we see that the formation of the link $ij$ is beneficial to individual $i$ only if his income shortfall resulting from a comparison of his income with the income of individual $j$ is smaller than his average income shortfall measured by his level of relative deprivation prior to the formation of the link. From (6) we see that the formation of the link $ij$ is beneficial to individual $j$ when prior to the formation he experienced relative deprivation. And if it so happens that both individuals have the same income, then by (5) we reach for individual $i$ the same conclusion as for individual $j$.

From (7) and (8) we see that when individuals $i$ and $j$ have several comparators, severance of the link $ij$ changes their relative deprivation in a manner that is the opposite of the manner presented by (5) and (6), respectively. In the special case in which individual $j$ is the only comparator of individual $i$, severing the link $ij$ leaves individual $i$ as an isolated node, relieving him of the experience of relative deprivation altogether. For individual $j$, it follows from (8) that if in the presence of the link $ij$ he experiences relative deprivation, then he would prefer to retain the connection with individual $i$ because doing so results in a larger comparison group, while his income shortfall remains at the same level (in comparison with the case in which $ij$ is severed), so individual $j$ stands to gain.

We next look at the effects of forming / severing the link $ij$ on the level of the index of income inequality in networks. The network obtained from $\mathcal{N} = (V, E)$ as a result of adding the link $ij$ is denoted by $\mathcal{N} \cup \{ij\}$, and the network obtained from severing the link $ij$ is denoted by $\mathcal{N} \backslash \{ij\}$. Because the formation / severance of the link $ij$ affects only the levels of relative deprivation of individuals $i$ and $j$, we see that:

1. Formation of the link $ij$ has the following consequence for the level of the index of income inequality in networks:

$$\Gamma(\mathcal{N} \cup \{ij\}, X) - \Gamma(\mathcal{N}, X) = \frac{RD(i, N_i \cup \{j\}) - RD(i, N_i) + RD(j, N_j \cup \{i\}) - RD(j, N_j)}{(n-1)\sum_{i=1}^{n} x_i}. \tag{9}$$

2. Severance of the link $ij$ has the following consequence for the level of the index of income inequality in networks:

$$\Gamma(\mathcal{N} \backslash \{ij\}, X) - \Gamma(\mathcal{N}, X) = \frac{RD(i, N_i \backslash \{j\}) - RD(i, N_i) + RD(j, N_j \backslash \{i\}) - RD(j, N_j)}{(n-1)\sum_{i=1}^{n} x_i}. \tag{10}$$

**Observation 4**. Consider two individuals $i$ and $j$ who, to begin with, are not connected. Then, the formation of the link $ij$ has the following consequence for the level of the index of income inequality in networks:

$$\Gamma(\mathcal{N} \cup \{ij\}, X) - \Gamma(\mathcal{N}, X) = \frac{(x_j - x_i) - \left( RD(i, N_i) + \frac{d_i + 1}{d_j + 1} RD(j, N_j) \right)}{(d_i + 1)(n-1)\sum_{i=1}^{n} x_i}.$$

We obtain this consequence by substituting (5) and (6) into (9).

From Observation 4 we infer that the formation of the link $ij$ decreases the level of the index of income inequality in networks if and only if the income shortfall of individual $i$, which he would have experienced if individual $j$ was his only comparator, is lower than the weighted sum of the levels of relative deprivation of individuals $i$ and $j$ prior to the formation of the link. In the last displayed equation, the weight of the first term, $RD(i, N_i)$, is 1, and the weight of the second term, $RD(j, N_j)$, is the ratio between the size of the comparison group of individual $i$ and the size of the comparison group of individual $j$, where both sizes are measured after the formation of $ij$. The effect of this formation is summarized in Figure 3.

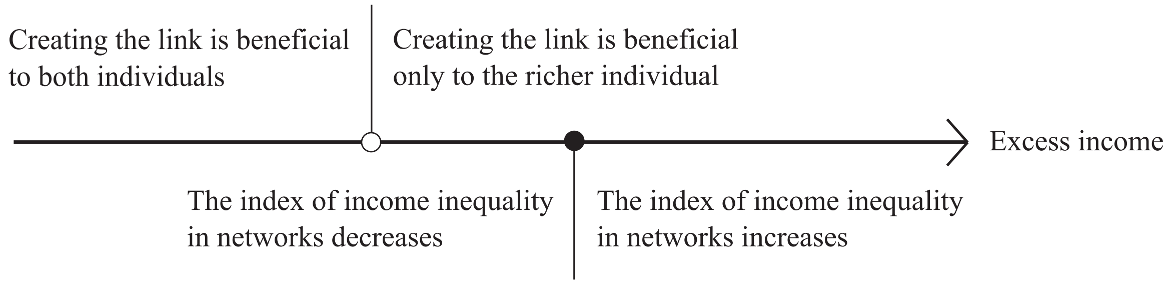


**Figure 3.** The consequences of creating the link $ij$. The empty circle marks the level $RD(i, N_i)$; the solid circle marks the level $RD(i, N_i) + \frac{d_i+1}{d_j+1}RD(j, N_j)$; and "Excess income" is the difference $x_j - x_i$.

Looking at (10) we see that after severing a link, the index $\Gamma(\mathcal{N}, X)$ decreases if and only if the sum of the changes of the levels of relative deprivation of individuals $i$ and $j$, that is, the numerator of (10), is negative. Whether or not the index decreases depends on the number of the comparators that individuals $i$ and $j$ have. For instance, when individual $i$ is the only comparator of individual $j$, severing the link $ij$ has no effect on the level of relative deprivation of individual $j$. In this case, either the index decreases unconditionally (as per case 1 below), or the decrease is conditioned by the change of the relative deprivation of individual $i$ (as per case 2 below).

1. If individuals $i$ and $j$ constitute each other's sole comparators ($d_i = d_j = 1$), then by (7), $RD(i, N_i\backslash\{j\}) - RD(i, N_i) = -(x_j - x_i)$, and by (8), $RD(j, N_j\backslash\{i\}) - RD(j, N_j) = 0$, so that the numerator in (10) reduces to $x_i - x_j \leq 0$, in which case the severance of $ij$ decreases the index $\Gamma(\mathcal{N}, X)$.
2. If individual $i$ has several comparators while he is the only comparator of individual $j$ ($d_i > 1$, $d_j = 1$), then the numerator of (10) reduces to $RD(i, N_i\backslash\{j\}) - RD(i, N_i) = \frac{1}{d_i - 1}\left[RD(i, N_i) - (x_j - x_i)\right]$, in which case the severance of $ij$ decreases the index $\Gamma(\mathcal{N}, X)$ if and only if as a result of the severance the relative deprivation of individual $i$ decreases.

## 5.2 Network characteristics

In this subsection we look at networks through the lens of network characteristics. As already noted, we refer to the average degree of a network, to the network density, and to the clustering coefficient.

**Claim 3**. Let the number of individuals and their incomes in network $\mathcal{N} = (V, E)$ be fixed. Assume that a connection between individuals $i, j \in V$, with $x_i \leq x_j$, is added only when the difference in the incomes of these individuals does not exceed the weighted sum of their levels of relative deprivation, that is, only when

$$x_j - x_i \leq RD(i, N_i) + \frac{d_i + 1}{d_j + 1}RD(j, N_j). \tag{11}$$

Then, because the average degree or the network density or the clustering coefficient of $\mathcal{N}$ increases, the level of the index of income inequality in $\mathcal{N}$ decreases.

**Proof**. The proof is in the Appendix.

Claim 3 builds on the results of Subsection 5.1 in that in a network with a fixed number of individuals, the only way to increase a given characteristic is through the addition of new links. Condition (11) ensures that the addition of a link that connects individuals $i$ and $j$ decreases the level of the index of income inequality in networks. Consequently, when a connection is created between two individuals with a difference in incomes that is small relative to the weighted sum of their levels of relative deprivation, our index decreases, while the three network characteristics (weakly) increase.

The next example illustrates why condition (11) is essential for obtaining Claim 3.

*Example 4*. Let $\mathcal{N}_1 = (V, E_1)$ be a network of three individuals, $V = \{1, 2, 3\}$, such that there are links between individuals 1 and 3 and between individuals 2 and 3, but initially there is no link between individuals 1 and 2. The income distribution is $X = (x_1, x_2, x_3)$: $x_1 = 10$, $x_2 = 4$, and $x_3 = 4$. Then

$$\Gamma(\mathcal{N}_1, X) = \frac{RD(1, \{3\}) + RD(2, \{3\}) + RD(3, \{1, 2\})}{2(10+4+4)}$$
$$= \frac{0 + 0 + \frac{1}{2}[(10-4) + (4-4)]}{36} = \frac{1}{12}.$$

The clustering coefficient of $\mathcal{N}_1$ is zero.

We denote by $\mathcal{N}_2 = (V, E_2)$ the network obtained from $\mathcal{N}_1 = (V, E_1)$ by adding a link between individuals 1 and 2. Because

$$x_1 - x_2 = 6 > 0 + \frac{2}{2}0 = RD(2, \{3\}) + \frac{d_2 + 1}{d_1 + 1} RD(1, \{3\}),$$

condition (11) is violated. And because $\mathcal{N}_2$ is a complete network of three individuals, the clustering coefficient of $\mathcal{N}_2$ is equal to 1. At the same time,

$$\Gamma(\mathcal{N}_2, X) = \frac{RD(1, \{2, 3\}) + RD(2, \{1, 3\}) + RD(3, \{1, 2\})}{2(10+4+4)} = \frac{\frac{1}{2}(0+6+6)}{36} = \frac{1}{6}.$$

Thus, while when transitioning from $\mathcal{N}_1$ to $\mathcal{N}_2$ the clustering coefficient increases, the level of the index of income inequality increases - it is higher in $\mathcal{N}_2$ than in $\mathcal{N}_1$.

**Corollary 1**. Let the number of individuals and their incomes in a network $\mathcal{N} = (V, E)$ be fixed. If the centrality of the position of the wealthiest individual (this individual's degree centrality) increases, then so does the level of the index of income inequality in networks.

This corollary follows from (11): by connecting with the wealthiest individual, we always obtain (11) but with the inequality sign reversed.

### 5.3 Configurations that maximize the index of income inequality in networks

We next ask which configurations of the individuals in a network maximize the index of income inequality in networks, assuming that while the individuals' incomes are fixed, the individuals can be repositioned.

**Claim 4**. Let the incomes of the individuals in network $\mathcal{N}$ be fixed. Then the configuration $(\mathcal{N}, X)$ maximizes the index of income inequality in the network if and only if: (i) any individual other than the richest individual in $(\mathcal{N}, X)$ has at least one connection; (ii) the set of individuals other than the richest in $(\mathcal{N}, X)$ is an independent set.

**Proof**. The proof is in the Appendix.

Claim 4 informs us that when a configuration of individuals maximizes the index of income inequality in a network, then the individuals other than the richest do not have connections between them. These individuals can be connected only with the richest individual(s). When several individuals are the richest, then there are no restrictions regarding the configuration of the links between them. In particular, a configuration that maximizes the index of income inequality in networks can be a star network in which a single individual (the richest) occupies the center of the star, or a bipartite network with several unconnected equally rich individuals.[7]

## 6. Conclusion

Having introduced measures of relative deprivation and total (aggregate) relative deprivation in networks, we constructed an index of income inequality in networks, tailor-made for this type of social architecture. We presented features of the index: we identified when the index attains a maximal value, and when it attains a minimal value; we studied the sensitivity of the index to transfers of income between members of the network; and we remarked that in a complete network the Gini coefficient can be derived from the index of income inequality in networks. We showed that the architecture of the network has a decisive impact on the total relative deprivation of the network and, consequently, on the level of income inequality, and we also showed how changes in the architecture of the network affect the income inequality in the network. We identified conditions under which the index decreases or increases when a new link is added or an existing link is severed. For example, we provided a condition under which an increase in the network density, that is to say, an increase in the number of links in the network, decreases the level of inequality in the network. We studied which income distributions with a fixed network architecture and which network architectures with a fixed income distribution result in the highest level of the index of income inequality in networks. And we also provided a sufficient condition for the Pigou-Dalton transfer principle to hold in a network setting.

Our approach provides a nuanced insight into the measurement of income inequality, and hints at a procedure by which a social planner could influence the extent of income inequality, not only by redistributing incomes, but also by reshaping the architecture of networks and their underlying social relations. Our inquiry in general, and our study of how changing the architecture of a network affects the level of income inequality of the network in particular, help to chart a path for studies of the dynamics of social networks.

## Appendix. Proofs of Proposition 1, and of Claims 1 through 4

For ease of reference, the proposition and claims proved in this appendix are replicated.

**Proposition 1**. Let $(\mathcal{N}, X)$ be an income network of $n$ individuals, $n \geq 3$. Then:
1. $0 \leq \Gamma(\mathcal{N}, X) \leq 1$.
2. $\Gamma(\mathcal{N}, X) = 0$ if and only if in every connected component of the network the component's income is divided equally between the component's members.

[7] In a bipartite network the nodes are divided into two disjoint groups. Then this network has connections between the groups but there are no connections within the groups.

3. $\Gamma(\mathcal{N}, X) = 1$ if and only if - a relabeling of the nodes notwithstanding - the income network $(\mathcal{N}, X)$ is identical to the income network $\Omega_n$.
4. In a complete income network $(\mathcal{N}_C, X)$, the Gini coefficient and the index $\Gamma(\mathcal{N}_C, X)$ can be obtained from each other by means of rescaling.

**Proof.**
For ease of exposition, we introduce the notation $x_{\min} = \min_{i \in V} x_i$, and $x_{\max} = \max_{i \in V} x_i$.
Part 1: $0 \leq \Gamma(\mathcal{N}, X) \leq 1$.
That $\Gamma(\mathcal{N}, X) \geq 0$ holds is obvious because the numerator and the denominator of $\Gamma(\mathcal{N}, X)$ are both non-negative. In order to show that $\Gamma(\mathcal{N}, X) \leq 1$, we look at the difference between the denominator ($\Gamma_D$) and the numerator ($\Gamma_N$) of (4):

$$\begin{aligned}\Gamma_D - \Gamma_N &= (n-1)\sum_{i=1}^{n} x_i - \sum_{i \in V^*}\left(\frac{1}{d_i}\sum_{j \in N_i}\max\{x_j - x_i, 0\}\right)\\ &= (n-1)\sum_{i=1}^{n} x_i - \sum_{\substack{i \in V^* \\ x_i < x_{\max}}}\left(\frac{1}{d_i}\sum_{\substack{j \in N_i \\ x_j > x_i}}(x_j - x_i)\right)\\ &= (n-1)\sum_{i=1}^{n} x_i - \sum_{\substack{i \in V^* \\ x_i < x_{\max}}}\left(\frac{1}{d_i}\sum_{\substack{j \in N_i \\ x_j > x_i}} x_j\right) + \sum_{\substack{i \in V^* \\ x_i < x_{\max}}}\left(\frac{1}{d_i}\sum_{\substack{j \in N_i \\ x_j > x_i}} x_i\right).\end{aligned} \tag{12}$$

From the fact that $\sum_{\substack{i \in V^* \\ x_i < x_{\max}}}\left(\frac{1}{d_i}\sum_{\substack{j \in N_i \\ x_j > x_i}} x_j\right) = \sum_{\substack{i \in V^* \\ x_i > x_{\min}}}\left(x_i\sum_{\substack{j \in N_i \\ x_j < x_i}}\frac{1}{d_j}\right)$, it follows from (12) that

$$\begin{aligned}\Gamma_D - \Gamma_N &= (n-1)\sum_{i=1}^{n} x_i - \sum_{\substack{i \in V^* \\ x_i > x_{\min}}}\left(x_i\sum_{\substack{j \in N_i \\ x_j < x_i}}\frac{1}{d_j}\right) + \sum_{\substack{i \in V^* \\ x_i < x_{\max}}}\left(\frac{1}{d_i}\sum_{\substack{j \in N_i \\ x_j > x_i}} x_i\right)\\ &= (n-1)\sum_{i \in V \setminus V^*} x_i + (n-1)\sum_{\substack{i \in V^* \\ x_i = x_{\min}}} x_i + \sum_{\substack{i \in V^* \\ x_i > x_{\min}}} x_i\left((n-1) - \sum_{\substack{j \in N_i \\ x_j < x_i}}\frac{1}{d_j}\right)\\ &\quad + \sum_{\substack{i \in V^* \\ x_i < x_{\max}}}\left(\frac{1}{d_i}\sum_{\substack{j \in N_i \\ x_j > x_i}} x_i\right).\end{aligned} \tag{13}$$

We note that if a node $i$ is not isolated, then $d_i \geq 1$. Therefore, $\sum_{\substack{j \in N_i \\ x_j < x_i}}\frac{1}{d_j} \leq n-1$. Combining this observation with our initial assumption that incomes are non-negative, we conclude from (13) that $\Gamma_D - \Gamma_N \geq 0$. Thus, $0 \leq \Gamma(\mathcal{N}, X) \leq 1$.
Part 2: $\Gamma(\mathcal{N}, X) = 0$ if and only if in every connected component of the network the component's income is divided equally between the component's members.

We note that $\Gamma(\mathcal{N}, X) = 0$ if and only if $TRD(\mathcal{N}, X) = 0$. Moreover, if $x_i \neq x_j$ for any $i$, $j$ members of the same connected component, then $TRD(\mathcal{N}, X) > 0$. Thus, $TRD(\mathcal{N}, X) = 0$ if and only if members of the same connected component have the same income.
Part 3: $\Gamma(\mathcal{N}, X) = 1$ if and only if $(\mathcal{N}, X)$ is identical to $\Omega_n$.
Because $\Gamma(\mathcal{N}, X) = 1$ if and only if $\Gamma_D = \Gamma_N$, it follows from (13) that

$$\Gamma(\mathcal{N}, X) = 1 \Leftrightarrow (n-1) \sum_{i \in V \setminus V^*} x_i + (n-1) \sum_{\substack{i \in V^* \\ x_i = x_{\min}}} x_i + \sum_{\substack{i \in V^* \\ x_i > x_{\min}}} x_i \left( (n-1) - \sum_{\substack{j \in N_i \\ x_j < x_i}} \frac{1}{d_j} \right)$$

$$+ \sum_{\substack{i \in V^* \\ x_i < x_{\max}}} \left( \frac{1}{d_i} \sum_{\substack{j \in N_i \\ x_j > x_i}} x_i \right) = 0.$$

Therefore, $\Gamma(\mathcal{N}, X) = 1$ if and only if the following conditions hold.

1. If $i$ is an isolated node, then $x_i = 0$.
2. $x_{\min} = 0$.
3. If $x_i > x_{\min}$, then $\sum_{\substack{j \in N_i \\ x_j < x_i}} \frac{1}{d_j} = n - 1$.
4. If $x_i < x_{\max}$ and $d_i > 0$, then $x_i = 0$.

From conditions 1, 2, and 4, almost all incomes, except the highest, are equal to zero. From condition 3, if an individual has an income that is not the lowest, then he has $n-1$ individuals in his comparison group, all of whom are poorer than he is, and for whom he is the only comparator. Because our characterization excludes isolated nodes, this configuration is possible only if $(\mathcal{N}, X)$ is a star network such that the central individual receives all the income.
Part 4: In a complete income network $(\mathcal{N}_C, X)$, the Gini coefficient and the index $\Gamma(\mathcal{N}_C, X)$ can be obtained from each other by means of rescaling.
We choose a representation of the Gini coefficient that can straightforwardly be applied to a complete network:

$$\bar{G}(X) = \frac{\frac{1}{n-1} \sum_{i=1}^{n} \sum_{j \in V \setminus \{i\}} \max\{x_j - x_i, 0\}}{\sum_{i=1}^{n} x_i}. \tag{14}$$

From computing the level of the index of income inequality of $(\mathcal{N}_C, X)$, we obtain

$$\Gamma(\mathcal{N}_C, X) = \frac{\sum_{i=1}^{n} \left( \frac{1}{n-1} \sum_{j \in V \setminus \{i\}} \max\{x_j - x_i, 0\} \right)}{(n-1) \sum_{i=1}^{n} x_i}. \tag{15}$$

Then, from (14) and (15) we get

$$\bar{G}(X) = \frac{\frac{1}{n-1}\sum_{i=1}^{n}\sum_{j\in V\setminus\{i\}}\max\{x_j - x_i, 0\}}{\sum_{i=1}^{n} x_i}$$

$$= (n-1)\frac{\sum_{i=1}^{n}\sum_{j\in V\setminus\{i\}}\max\{x_j - x_i, 0\}}{(n-1)^2\sum_{i=1}^{n} x_i} = (n-1)\Gamma(\mathcal{N}_C, X).$$

Q.E.D.

**Claim 1**. Let $y > 0$ be the aggregate income of network $\mathcal{N} = (V, E)$, and let $I$ be the set of all the individuals for whom the term $\sum_{j\in N_i}\frac{1}{d_j}$ is maximal. We then choose any nonempty independent set $J \subset I$. Then, the distribution of the network's income $y$, where $y$ is divided (in any manner) between the members of the set $J$, maximizes the index $\Gamma(\mathcal{N}, X)$. In particular, if $|I| = 1$, then $\Gamma(\mathcal{N}, X)$ records its maximal value when this one member of $I$ receives all the income of the network.

**Proof**.
Given the aggregate income of the network, the denominator of the index $\Gamma(\mathcal{N}, X)$ in (4) does not depend on the income distribution of the network. Thus, we have analyzed the numerator of $\Gamma(\mathcal{N}, X)$, that is, the total relative deprivation of the network.

Let $y$ be the aggregate income of the network. From formula (2) of the total relative deprivation of an income network, we obtain

$$\begin{aligned} TRD(\mathcal{N}, X) &= \sum_{i\in V^*}\left(\frac{1}{d_i}\sum_{j\in N_i}\max\{x_j - x_i, 0\}\right) \le \sum_{i\in V^*}\left(\frac{1}{d_i}\sum_{j\in N_i} x_j\right) = \sum_{i\in V^*}\left(x_i\sum_{j\in N_i}\frac{1}{d_j}\right) \\ &\le \sum_{i\in V}\left(x_i \max\left\{\sum_{j\in N_i}\frac{1}{d_j} : i\in V^*\right\}\right) = \left(\sum_{i\in V} x_i\right)\cdot\max\left\{\sum_{j\in N_i}\frac{1}{d_j} : i\in V^*\right\} \\ &= y\cdot\max\left\{\sum_{j\in N_i}\frac{1}{d_j} : i\in V^*\right\}. \end{aligned} \tag{16}$$

From (16), it follows that for every income distribution of income network $(\mathcal{N}, X)$, the level of $TRD(\mathcal{N}, X)$ is bounded from above by $y\cdot\max\left\{\sum_{j\in N_i}\frac{1}{d_j} : i\in V^*\right\}$.

Let $I \equiv \arg\max\left\{\sum_{j\in N_i}\frac{1}{d_j} : i\in V^*\right\}$. We then choose a nonempty independent set $J \subset I$. We assume that $J$ has $k \ge 1$ elements, that is, $J = \left\{i_1^*, i_2^*, ..., i_k^*\right\}$. We allocate the aggregate income of the network between the members of $J$ so that $x_{i_1^*} + x_{i_2^*} + \ldots + x_{i_k^*} = y$. Then, all the neighbors of the individuals from $J$ (and only these individuals) experience relative deprivation. The level of *TRD* of this network is

$$TRD(\mathcal{N}, X) = \sum_{l \in J} \sum_{j \in N_l} \frac{1}{d_j}(x_l - 0) = \sum_{l \in J}\left(x_l \sum_{j \in N_l} \frac{1}{d_j}\right) = \left(\sum_{l \in J} x_l\right)\left(\sum_{j \in N_l} \frac{1}{d_j}\right) = y \sum_{j \in N_l} \frac{1}{d_j}. \tag{17}$$

The penultimate equality in (17) follows from the feature that for the members of $J$ (and $I$), all the expressions $\sum_{j \in N_l} \frac{1}{d_j}$ for $l \in J$ are equal to each other. From the definition of the set $I$, we obtain that the level of $TRD(\mathcal{N}, X)$ in (17) is equal to the upper bound from (16). Therefore, by allocating (in any manner) the aggregate income of the network to the members of the set $J$, we obtain the maximal levels of $TRD(\mathcal{N}, X)$ and $\Gamma(\mathcal{N}, X)$.

For the case in which $I$ has only one element, its only nonempty independent subset is $J = I$. Then, the levels of $TRD(\mathcal{N}, X)$ and $\Gamma(\mathcal{N}, X)$ are maximal when the aggregate income of the network is allocated to the single member of $I$.

Q.E.D.

**Claim 2**. Consider an income network $(\mathcal{N}, X)$ of $n$ individuals, $n \geq 3$. We assume a top-down (respectively, a bottom-up) rank-preserving transfer of income from individual $i$ to individual $k$. Then, the Pigou-Dalton principle holds, that is, the level of income inequality decreases (respectively, increases) when the following inequality is satisfied:

$$\sum_{\substack{l \in N_k \\ x_l > x_k}} \frac{1}{d_k} - \sum_{\substack{l \in N_k \\ x_l < x_k}} \frac{1}{d_l} \underset{(<)}{>} \sum_{\substack{j \in N_i \\ x_j > x_i}} \frac{1}{d_i} - \sum_{\substack{j \in N_i \\ x_j < x_i}} \frac{1}{d_j}.$$

**Proof**.
We note that because a transfer of income between individuals does not affect the denominator of (4), we analyze the numerator of (4), that is, the *TRD* of the network.

We consider a rank-preserving transfer of $\varepsilon > 0$ units of income from individual $i$ to individual $k$. Then, only the neighbors of individual $i$ who are poorer than himself experience an income shortfall from comparisons of their incomes with the income of individual $i$. Prior to the transfer, the income shortfall of individual $j \in N_i$ who is poorer than individual $i$ is $x_i - x_j$, and following the transfer, the income shortfall of individual $j$ is $x_i - \varepsilon - x_j$. Thus, the transfer decreases the relative deprivation of individual $j$ by $\frac{\varepsilon}{d_j}$.

At the same time, the level of relative deprivation of individual $i$ increases by $\sum_{\substack{j \in N_i \\ x_j > x_i}} \frac{\varepsilon}{d_i}$. By aggregating over individual $i$ and his neighbors, we find that the transfer decreases the sum of the levels of relative deprivation of these individuals by $\sum_{\substack{j \in N_i \\ x_j < x_i}} \frac{\varepsilon}{d_j} - \sum_{\substack{j \in N_i \\ x_j > x_i}} \frac{\varepsilon}{d_i}$.

By performing an analogous reasoning for individual $k$, we see that the transfer decreases the sum of the levels of relative deprivation of individual $k$ and his neighbors by $\sum_{\substack{l \in N_k \\ x_l > x_k}} \frac{\varepsilon}{d_k} - \sum_{\substack{l \in N_k \\ x_l < x_k}} \frac{\varepsilon}{d_l}$.

In sum, the top-down (bottom-up) transfer decreases (increases) the *TRD* of the network if

$$\sum_{\substack{j\in N_i \\ x_j<x_i}} \frac{\varepsilon}{d_j} - \sum_{\substack{j\in N_i \\ x_j>x_i}} \frac{\varepsilon}{d_i} + \sum_{\substack{l\in N_k \\ x_l>x_k}} \frac{\varepsilon}{d_k} - \sum_{\substack{l\in N_k \\ x_l<x_k}} \frac{\varepsilon}{d_l} \underset{(<)}{>} 0.$$

This inequality is equivalent to

$$\sum_{\substack{l\in N_k \\ x_l>x_k}} \frac{1}{d_k} - \sum_{\substack{l\in N_k \\ x_l<x_k}} \frac{1}{d_l} \underset{(<)}{>} \sum_{\substack{j\in N_i \\ x_j>x_i}} \frac{1}{d_i} - \sum_{\substack{j\in N_i \\ x_j<x_i}} \frac{1}{d_j}.$$

Q.E.D.

**Claim 3**. Let the number of individuals and their incomes in network $\mathcal{N} = (V, E)$ be fixed. Assume that a connection between individuals $i, j \in V$, with $x_i \le x_j$, is added only when the difference in the incomes of these individuals does not exceed the weighted sum of their levels of relative deprivation, that is, only when

$$x_j - x_i \le RD(i, N_i) + \frac{d_i + 1}{d_j + 1} RD(j, N_j). \tag{18}$$

Then, because the average degree or the network density or the clustering coefficient of $\mathcal{N}$ increases, the level of the index of income inequality in $\mathcal{N}$ decreases.

**Proof**.
Because the number of individuals in $\mathcal{N} = (V, E)$ is fixed, the following observations hold.
(i) The only method to increase the average degree of $\mathcal{N}$ or the network density is to add new connections.
(ii) The clustering coefficient increases when the number of complete subnetworks of three individuals increases, and this can be obtained only by adding new connections.
(iii) From the results of Subsection 5.1 (recalling Figure 3), we infer from condition (18) that the addition of a new connection to the network $\mathcal{N} = (V, E)$ decreases the level of the index of income inequality in networks.

Q.E.D.

**Claim 4**. Let the incomes of the individuals in network $\mathcal{N}$ be fixed. Then the configuration $(\mathcal{N}, X)$ maximizes the index of income inequality in the network if and only if: (i) any individual other than the richest individual in $(\mathcal{N}, X)$ has at least one connection; (ii) the set of individuals other than the richest in $(\mathcal{N}, X)$ is an independent set.

**Proof**.
We prove the claim by contradiction.

First, we assume that in the configuration that maximizes the index of income inequality in networks there is one individual who is not the richest and who is at an isolated node. Then, by connecting this individual with one of the richest individuals, we increase the level of relative deprivation of the former individual (the levels of relative deprivation of the other individuals remain unchanged). Therefore, following this operation, the level of the index of income inequality in networks increases, which contradicts the assumption that this configuration maximizes the index.

Second, we assume that in the configuration that maximizes the index of income inequality in networks the set of individuals other than the richest is not an independent set. Thus, we can consider the poorest individual $i$ who has a link to another individual who is not the richest. We denote the maximal income in this network by $x_{\max}$. Then

$$RD(i, N_i) = \frac{1}{d_i}\sum_{k\in N_i}\max\{x_k - x_i, 0\} < \frac{1}{d_i}\sum_{k\in N_i}\max\{x_{\max} - x_i, 0\} = x_{\max} - x_i. \quad (19)$$

From (19), we infer that deleting all the existing links of individual $i$ and connecting him only with one of the richest individuals increases his level of relative deprivation. Moreover, following this operation, the levels of relative deprivation of all the comparators of individual $i$ will also (weakly) increase (the levels of relative deprivation of the other individuals will not change). As a result, the level of the index of income inequality in networks increases, which contradicts the assumption that the configuration presented at the beginning of this paragraph maximizes the index.

Q.E.D.

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